Final Exam of Quantum Physics 1 – 2023/2024

Thursday, November 2, 2023, 18:15 – 20:15

Read these instructions carefully. If you do not follow them your exam might be (partially) voided.

- This exam consists of 3 questions in 2 pages and a formula sheet at the end.
- The points for each question are indicated on the left side of the page.
- You have 2 hours to complete this exam.
- **Write your name and student number on *all* answer sheets that you turn in**.
- Start answering each exercise on a new page. It is ok to use front and back.
- Clearly write the total number of answer sheets that you turn in on the first page.
- Telephones, smart devices, and other electronic devices are **NOT** allowed.
- **This is a closed book exam**. Consulting reading material is **not** allowed.

40 pts

Question 1

For this question, consider an electron subject to the following potential:

$V(x) = \begin{cases} -V_0, & \text{for} \quad -a < x < a \\ 0, & \text{otherwise} \end{cases}$, where V_0 and a are positive constants.

We will start by considering the limit where V_0 is very large ($V_0 \to \infty$) and that the electron only occupies (negative) energies just above V_0.

3 pts **a)** With the assumptions above, sketch the wavefunction for the ground and first excited state.

10 pts **b)** Give the explicit forms for the wavefunctions of the stationary states and their energies. You do not need to normalize them. *Tip*: You can solve the Schrodinger equation or use arguments to support your answer. If you solve the Schrodinger equation, you only need to solve for one or two *n* and argue why it will be the same (or will change) for the other states.

12 pts **c)** The interaction of the electron with light can be understood using the dipole moment operator $\widehat{D} = ex$, where *e* is the electron charge. Strong dipole oscillations occur when the system is in a superposition of two different energy eigenstates $|\phi_m\rangle$ and $|\phi_n\rangle$. Show that if the system is in a superposition of two eigenstates – i.e. $|\alpha(t)\rangle = a|\phi_m(t)\rangle + b|\phi_n(t)\rangle$ – the system cannot emit a photon ($\langle\widehat{D}\rangle$=0). *Hint*: Use the *x*-representation to evaluate all the elements like $\langle\phi_n(t)|\widehat{D}|\phi_m(t)\rangle$ in an expression for how the dipole moment oscillates as a function of time during the emission of a photon.

3 pts **Now we will get rid of the assumption that V_0 is very large, *i.e.* we will assume a *shallow* potential.**

d) Make a sketch of the wavefunction for the ground and first excited state for this case. Indicate clearly what has changed in comparison to your answer in b).

Now consider that the electron is not entrapped by this potential, *i.e.*, the electron is *free,* and its total energy is positive.

6 pts **e)** Give the general forms for the wavefunction for the three regions: Region I: $x < -a$, Region II: $-a < x < a$, and Region III: $x > a$. Assume that the particle travels from left ($x < 0$) to right ($x > 0$). **Attention!** Do not bother normalizing the wavefunctions. You do not need to apply the boundary conditions either.

6 pts **f)** A special phenomenon occurs when half the De Broglie wavelength of the free electron in Region II is equal to a multiple of *2a*, i.e., when $n(\lambda/2) = 2a$. What happens when this condition is met? Calculate the energy equivalent for these values of wavelength. Compare your result to the answer you found for the energy in **c)**.

<u>30 pts</u> **Question 2**

When an electron is subjected to a magnetic field (***B***), the Hamiltonian of the system is given by:

$H = -\gamma \boldsymbol{S} \cdot \boldsymbol{B}$,

where γ is the gyromagnetic ratio constant and $\gamma > 0$, and $\boldsymbol{S} = (S_x,\ S_y,\ S_z)$ is the spin operator.

5 pts **a)** If $\boldsymbol{B} = B_0\,\hat{\boldsymbol{z}}$ and B_0 is constant (and positive), write the Hamiltonian explicitly in its matrix form and find the eigenvalues and energy eigenvectors of the Hamiltonian above.

10 pts **b)** Suppose that we prepare the system in its ground state with the Hamiltonian above and then quickly, much faster than the system can respond, rotate the field towards the *x*-direction: $\boldsymbol{B} = B_0\,\hat{\boldsymbol{x}}$. What is the time dependence of the state in the basis of this new Hamiltonian (basis of S_x)? *Hint*: You can get the eigenvectors of S_x from its matrix: $S_x = \frac{\hbar}{2}\sigma_x$.

7 pts **c)** Calculate the effect of the operator S_z on each of the eigenvectors of S_x. In other words, what is $S_z|+\rangle_x$ and $S_z|-\rangle_x$?

8 pts **d)** What is the (time-dependent) expectation value for S_x and S_z?

<u>30 pts</u> **Question 3**

Consider the one-dimensional harmonic oscillator, with potential $V(x) = \frac{1}{2}m\omega^2x^2$, where m is the mass of the particle trapped in the potential and ω the natural frequency. Using the ladder operators $\hat{a}_+$ and $\hat{a}_-$, we can rewrite the Hamiltonian for this system as $H = \hbar\omega\left(\hat{a}_\pm\hat{a}_\mp \pm \frac{1}{2}\right)$, for which *n* stationary states ψ_n are found.

The operators $\hat{x}$ and $\hat{p}$ can be written in terms of the ladder operators: $\hat{a}_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp i\,\hat{p} + m\omega x)$, as $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)$ and $\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-)$.

3 pts **a)** Using the relations $\hat{a}_+\psi_n = \sqrt{n+1}\,\psi_{n+1}$ and $\hat{a}_-\psi_n = \sqrt{n}\,\psi_{n-1}$, calculate $\hat{a}_+\hat{a}_-\psi_n$ and $\hat{a}_-\hat{a}_+\psi_n$, where ψ_n are the energy eigenstates of the harmonic oscillator.

8 pts **b)** Use the relations above to calculate the expectation values $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, and $\langle p^2\rangle$ for the state ψ_n. Show that they obey the Heisenberg uncertainty principle.

7 pts **c)** Use the fact that $\hat{a}_-\psi_0 = 0$ to find the ground state wave function ψ_0. You do not have to normalize it.

7 pts **d)** Find the first excited state using the raising operator. You do not have to normalize it.

2 pts **e)** Sketch the ground state and the first excited state.

3 pts **f)** Now consider the half harmonic oscillator:

$V(x) = \frac{1}{2}m\omega^2x^2$ for $x > 0$ and $V(x) = \infty$ for $x \leq 0$.

Argue what the ground state of the system is using symmetry arguments. What about the excited states?

Useful formulas:

Schrodinger equation	$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi$
Time-independent Schrodinger equation	$H\psi = E\psi \qquad \Psi = \psi e^{-iEt/\hbar}$
Hamiltonian operator	$H = -\frac{\hbar^2}{2m}\nabla^2 + V$
Momentum operator	$p = -i\hbar\,\nabla$
De Broglie wavelength	$\lambda = h/p$
Time-dependence of expectation value	$\frac{d\langle Q\rangle}{dt} = \frac{i}{\hbar}\langle[H,Q]\rangle + \langle\frac{\partial Q}{\partial t}\rangle$
Generalized uncertainty principle	$\sigma_A\sigma_B \geq \left\lvert\frac{1}{2i}\langle[A,B]\rangle\right\rvert$
Heisenberg Uncertainty principle	$\sigma_x\sigma_p \geq \hbar/2$
Canonical commutator	$[x,p] = i\hbar$
Angular momentum	$[L_x, L_y] = i\hbar L_z \;;\; [L_y, L_z] = i\hbar L_x \;;\; [L_z, L_x] = i\hbar L_y$
Pauli matrices	$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \;;\; \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \;;\; \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

Trigonometric relations

$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$

$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$

$\sin(90^\circ \pm \theta) = \cos(\theta)$

$\cos(90^\circ \pm \theta) = \mp\sin(\theta)$